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ON STABILITY OF A PLANE-PARALLEL CONVECTIVE FLOW OF A BINARY MIXTURE*

G. Z. GERSHUNI, E. M. ZHUKHOVITSKII and L. E. SOROKIN

The problem of stability of the convective flow of a binary mixture in a plane vertical layer is considered in the same formulation as in /1/. Numerical and analytical methods are used for solving the spectral amplitude problem. The investigation shows that in different domains of parameters instability depends on different mechanisms. It is shown that, unlike in /1/, in the region of thermal concentration destabilization the crisis of flow is due to long-wave perturbations.

Stability of the convective flow of a binary mixture was investigated in /1/ without allowance for thermal diffusion in the presence of longitudinal thermally stable stratification. Existence of a substantial decrease of flow stability in the region of finite values of the stratification parameter, due to the heat concentration mechanism, was disclosed. However the flow data on flow destabilization are erroneous, as will be shown below. The layered convective flows of a mixture induced by thermal concentration instability were investigated in a number of publications (see the survey in /2/). The thermal concentration instability mechanism of filtration of a mixture through a porous medium was studied in /3/. The stability of flow in a vertical layer was considered in /4/, with transverse temperature differences and the concentration difference at boundaries taken into account.

1. Statement of the problem.

A plane vertical layer of a binary mixture bounded by solid parallel walls at $x=\pm h$ impermeable to the substance, at which different constant temperatures $\pm\Theta$ are maintained, is considered. The vertical concentration gradient that corresponds to the potentially stable stratification in the fluid is specified. Under such conditions mechanical stability is possible, and flow is generated. We define the flow using the equations of mixture convection in the Boussinesq approximation, disregarding thermal diffusion and diffusion heat conduction. We introduce dimensionless variables using the following units: h for distance, h^2/ν for time, $g\beta_1\Theta h^2/\nu$ for velocity, Θ for temperature, $\beta_1\Theta/\beta_2$ for concentration, and $\rho_0 g\beta_1\Theta h$ for pressure. We have the following dimensionless equations and conditions at the layer boundaries, of the stream closure and of constancy of the vertical gradient of concentration:

$$\frac{\partial \mathbf{v}}{\partial t}+G(\mathbf{v}\nabla)\mathbf{v}=-\nabla p+\Delta\mathbf{v}+(T+C)\boldsymbol{\gamma},\quad \nabla\mathbf{v}=0,\quad \frac{\partial T}{\partial t}+G(\mathbf{v}\nabla T)=\frac{1}{P}\Delta T,\quad \frac{\partial C}{\partial t}+G(\mathbf{v}\nabla C)=\frac{1}{P_d}\Delta C \tag{1.1}$$

$$x=\pm 1,\ \mathbf{v}=0,\quad T=\pm 1,\quad \partial C/\partial x=0,\quad \int_{-1}^{1} v_z\,dx=0,\quad \lim_{l\to\infty}\frac{1}{2l}\int_{-l}^{l}\frac{\partial C}{\partial z}\,dz=\frac{R_d}{GP_d} \tag{1.2}$$

$$G=\frac{g\beta_1\Theta h^3}{\nu^2},\quad R_d=\frac{g\beta_2 Bh^4}{\nu D},\quad P=\frac{\nu}{\chi},\quad P_d=\frac{\nu}{D}$$

where p is the convection addition to the hydrostatic pressure, C is the concentration of the light component, and T is the temperature, all measured from some /initial/ values, β_1 and β_2 are temperature and concentration density coefficients, γ is a unit vector on the vertical z-axis upward, and the remaining symbols are of conventional form.

Problem (1.1), (1.2) contains four dimensionless parameters: the Grashof number G determined by the transverse temperature difference, the concentration Rayleigh number R_d determined by the longitudinal gradient of concentration B, and the usual and the diffusion Prandtl numbers P and P_d, respectively.

The problem thus formulated has a stationary solution which defines the plane-parallel flow in a fairly long layer. The respective distributions of velocity, temperature, and concentration are

$$v_0=\frac{1}{2N\mu^3}\left(\frac{\operatorname{sh}\mu x\cos\mu x}{\operatorname{sh}\mu\cos\mu}-\frac{\operatorname{ch}\mu x\sin\mu x}{\operatorname{ch}\mu\sin\mu}\right),\quad T_0=x,\quad C_0=\frac{R_d}{GP_d}z-x+\frac{1}{N\mu}\left(\frac{\operatorname{sh}\mu x\cos\mu x}{\operatorname{ch}\mu\sin\mu}+\frac{\operatorname{ch}\mu x\sin\mu x}{\operatorname{sh}\mu\cos\mu}\right) \tag{1.3}$$

$$N=\operatorname{tg}\mu+\operatorname{ctg}\mu-\operatorname{th}\mu+\operatorname{cth}\mu;\quad \mu=\left(\frac{R_d}{4}\right)^{1/4}$$

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We begin the investigation of linear stability of the stationary flow (1.3) by considering the small plane normal perturbations

$$(\psi', T', C') = (\varphi, \theta, \xi) \exp(-\lambda t + ikz) \tag{1.4}$$

where ψ' is the stream function of the velocity field perturbation, $\varphi(x)$, $\theta(x)$, $\xi(x)$ are amplitudes, k is the wave number, and $\lambda = \lambda_r + i\lambda_i$ is the perturbation decrement. Linearization of Eqs. (1.1) near the stationary solution (1.3) yields the system of amplitude equations with homogeneous boundary conditions

$$-\lambda(\varphi'' - k^2\varphi) + ikG[v_0(\varphi'' - k^2\varphi) - v_0''\varphi] = (\varphi^{IV} - 2k^2\varphi'' + k^4\varphi) + \theta' + \xi' \tag{1.5}$$

$$-\lambda\theta + ikG(v_0\theta - T_0'\varphi) = \frac{1}{P}(\theta'' - k^2\theta), \quad -\lambda\xi + ikG(v_0\xi - C_0'\varphi) + \frac{R_d}{P_d}\varphi' = \frac{1}{P_d}(\xi'' - k^2\xi)$$

$$x = \pm 1, \quad \varphi = \varphi' = 0, \quad \theta = 0, \quad \xi' = 0 \tag{1.6}$$

Here and subsequently a prime denotes differentiation with respect to the transverse coordinate x.

The spectral amplitude problem (1.5), (1.6) determines the eigenvalues which represent the characteristic decrements λ dependent on all parameters G, R_d, P, P_d, k and, also, the respective eigenfunctions which represent perturbation amplitudes. The stability limit for monotonic type perturbations defined by real λ are obtained from the condition $\lambda = 0$, and the neutral mode of oscillating perturbations from the condition $\lambda_r = 0$.

2. **Long-wave perturbations.** By analogy with the results of stability investigations of convective filtration of a mixture in a vertical porous layer /3/, it can be expected that in the important range of parameter variation, the most dangerous will be the long-wave perturbations as $k \to 0$. To solve the boundary value problem (1.5), (1.6) in this limit case we use the method of the small parameter.

Setting in (1.5) $k = 0$ we obtain the equation

$$-\lambda\varphi'' = \varphi^{IV} + \theta' + \xi', \quad -\lambda P\theta = \theta'', \quad -\lambda P_d\xi + R_d\varphi' = \xi'' \tag{2.1}$$

with boundary conditions (1.6). All perturbations determined by this boundary value problem are damped, except the unique level (of the concentration type) in the spectrum which is neutral

$$\lambda = 0, \quad \varphi = \theta = 0, \quad \xi = \text{const} \tag{2.2}$$

(below, normalization is carried out with const $= 1$). The perturbation that corresponds to that level may prove to be (and actually proves to be) increasing for small k in a particular domain of parameters, since it is neutral when $k = 0$. For the investigation of stability with respect to such long-wave perturbation we represent the solution in the form of expansion in the small parameter k

$$\varphi = \varphi_1 k + \varphi_2 k^2 + \ldots, \quad \theta = \theta_1 k + \theta_2 k^2 + \ldots, \quad \xi = 1 + \xi_1 k + \xi_2 k^2 + \ldots; \quad \lambda = \lambda_1 k + \lambda_2 k^2 + \ldots \tag{2.3}$$

In the first order with respect to k we have the system of equations

$$\varphi_1^{IV} + \theta_1' + \xi_1' = 0, \quad \theta_1'' = 0, \quad \xi_1'' - R_d\varphi_1' = -\lambda_1 P_d + iGP_d v_0 \tag{2.4}$$

(since the boundary conditions for all orders coincide with (1.6), they will not, henceforth, be written out). The condition of solvability of the inhomogeneous system (2.4) is that $\lambda_1 = 0$, and the solution is of the form

$$\varphi_1 = \frac{iGP_d}{32N\mu^7}\left(A_0 + A_1 \operatorname{ch}\mu x \cos\mu x + A_2 \operatorname{sh}\mu x \sin\mu x + \frac{x \operatorname{sh}\mu x \cos\mu x}{\operatorname{sh}\mu \cos\mu} - \frac{x \operatorname{ch}\mu x \sin\mu x}{\operatorname{ch}\mu \sin\mu}\right) \equiv iGP_d\bar{\varphi}_1 \tag{2.5}$$

$$\xi_1 = \frac{iGP_d}{16N\mu^5}(B_1 \operatorname{ch}\mu x \sin\mu x + B_2 \operatorname{sh}\mu x \cos\mu x + B_3 x \operatorname{sh}\mu x \sin\mu x + B_4 x \operatorname{ch}\mu x \cos\mu x) \equiv iGP_d\bar{\xi}_1, \quad \theta_1 = 0$$

$$A_0 = \frac{2N}{\mu}, \quad A_1 = -\frac{2\operatorname{sh}\mu \sin\mu}{\operatorname{sh}2\mu + \sin 2\mu}\left[N_1 + \frac{2N}{\mu}(\operatorname{cth}\mu + \operatorname{ctg}\mu)\right], \quad A_2 = \frac{2\operatorname{ch}\mu\cos\mu}{\operatorname{sh}2\mu + \sin 2\mu}\left[N_1 + \frac{2N}{\mu}(\operatorname{th}\mu - \operatorname{tg}\mu)\right]$$

$$B_1 = \mu(A_1 + A_2) + \frac{3}{\operatorname{sh}\mu\cos\mu}, \quad B_2 = \mu(A_1 - A_2) + \frac{3}{\operatorname{ch}\mu\sin\mu}, \quad B_3 = \mu\left(\frac{1}{\operatorname{sh}\mu\cos\mu} - \frac{1}{\operatorname{ch}\mu\sin\mu}\right), \quad B_4 = \mu\left(\frac{1}{\operatorname{sh}\mu\cos\mu} + \frac{1}{\operatorname{ch}\mu\sin\mu}\right)$$

$$N_1 = \operatorname{tg}\mu + \operatorname{ctg}\mu + \operatorname{th}\mu - \operatorname{cth}\mu$$

For second order amplitudes we obtain the system

$$\varphi_2^{IV} + \theta_2' + \xi_2' = iG(v_0\varphi_1'' - v_0''\varphi_1), \quad \theta_2'' = -iGPT_0'\varphi_1, \quad \xi_2'' - R_d\varphi_2' = 1 + iG(v_0\xi_1 - C_0'\varphi_1) - \lambda_2 \tag{2.6}$$

whose solvability condition consists of the requirement that the integral in the right-hand side of the last equation must vanish. This determines the first nonvanishing term in the expansion of the decrement

$$\lambda = \lambda_2 k^2 + \ldots, \quad \lambda_2 = \frac{1}{P_d}\left(1 - \frac{G^2}{G_c^2}\right) \tag{2.7}$$

$$G_c = \frac{F(\mu)}{P_d}, \quad F(\mu) = \left[\int_0^1 (v_0 \bar{\xi}_1 - C_0' \bar{\varphi}_1)\, dx\right]^{-1/2} \tag{2.8}$$

It is evident from (2.7) that the decrement λ in region $G < G_c$ is positive when $k \neq 0$, i.e. the long-wave perturbations die out when k is small. In region $G > G_c$ the decrement is negative when k is small, i.e. we have long-wave instability. The value of G_c determined by formula (2.8) defines the stability limit with respect to long-wave perturbations. The critical parameter is determined by the product $G_c P_d$ which depends on the longitudinal stratification parameter $\mu = (R_d/4)^{1/4}$. Owing to the unwieldiness of the expression for function $F(\mu)$, it is not given here. The curve of $F(\mu)$ is shown in Fig.1, where the considered instability mode appears to the right of asymptote $\mu > \mu_* = 1.673$, respectively $R_d > R_{d*} = 31.3$. The minimum stability of the stream corresponds to the stratification parameter $\mu_m = 1.989$, i.e. $R_{dm} = 62.6$, and the critical Grashof number is then $G_c = 196.12 / P_d$.

3. Numerical results. The boundary value problem (1.5), (1.6) was solved numerically for finite wave numbers k.

Three methods were used: that of Galerkin and those of Runge—Kutta with step-by-step orthogonalization, and of differential runs. For the approximation of perturbation amplitudes of functions of stream and temperature we used the Galerkin method on the same bases as used in solving stability problems of homogeneous fluid flow /5/, which corresponds to perturbations in a stationary fluid. For approximating the amplitudes of concentration perturbations we used eigenfunctions, normalized in a specific way, of the boundary value problem

$$\xi'' - k^2\xi = -\lambda P_d \xi, \quad \xi'(\pm 1) = 0$$

The number of basis functions retained in expansions of φ, θ, and ξ was determined by the requirement for intrinsic convergence of the method it was, also, dependent on parameters of the problem. Numerical integration by the Runge—Kutta—Merson method with orthogonalization was carried out in conformity with the scheme described in /6/ in connection with problems of convective flow stability. Numerical integration of amplitude equations was carried out by the method of differential runs /7/.

The spectrum of eigenvalues λ, the stability limit, and parameters of critical perturbations were obtained from the solution. The Galerkin method was mainly used for obtaining an over-all picture of the decrement spectrum, while the determination of stability limits and of parameters of critical perturbations was in the main obtained using the two numerical methods mentioned above. All of these methods yielded concurrent results within the range of parameters considered here.

Some of the results of calculations are shown in Figs. 2—5. The dependence of the lowest (with respect to k) critical Grashof number G_m on the concentration Rayleigh number R_d for fixed P and P_d is shown in Fig.2 (the curves correspond to the following combinations of parameters: 1 for $P = 6.7$, $P_d = 676.7$, 2 for $P = 6.7$, $P_d = 100$, 3 for $P = 6.7$, $P_d = 30$, and 4 for $P = 0.7$, $P_d = 1.3$).

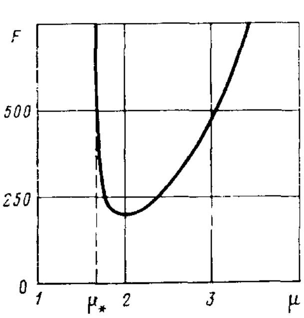

Fig.1

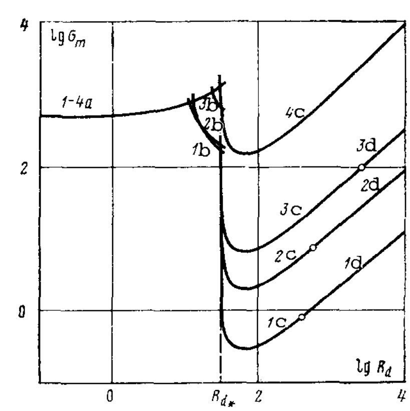

Fig.2

The flow in various regions of variation of the basic parameter R_d shows instability which is due to various mechanisms. As an example, let us consider the stability limit for $P = 6.7$, $P_d = 676.7$ (a typical example of a fluid binary mixture, such as water solution of salt). At low R_d $(0 < R_d < 13$, shown by curve *1a* the instability is of purely hydrodynamic nature, and in a homogeneous fluid $(R_d = 0)$ is associated with the formation of stationary vortices at the interface of counterflows. The increase of the vertical concentration gradient has some stabilizing effect. Owing to the hydrodynamic nature of the crisis, the stability limit in that region weakly depends on P and P_d (sections of curves *1a — 4a* are the same within the accuracy of the diagram).

In region $13 < R_d < 30$ (curve *1*b) the most dangerous is the wave mode. That mode is associated with increasing oscillatory perturbations of concentration, with two equivalent, from the point of view of wave stability, convection streams propagating up- and downward.

For $R_d > R_{d*} \approx 30$ stability is of the thermal-concentration nature and substantially depends on the presence of vertical stratification of the mixture. In region $R_{d*} < R_d < R_{d0}$ (for the considered values of parameters $R_{d0} = 399$) this instability is associated with stationary long-wave perturbations ($k = 0$, section of curve *1*c) which are the most dangerous. Numerical values of the critical Grashof number of long-wave instability fully coincide, for all P and P_d, with those obtained in the asymptotic analysis in Sect.2, formula (2.8). These results show that the thermal concentration mode leads to considerable destabilization of the flow. This is particularly strongly evident at high P_d (liquid solutions); thus the critical Grashof number along curve *1* is by three orders lower than in the region of action of the hydrodynamic instability mechanism.

For $R_d > R_{d0}$, where R_{d0} depends on P and P_d, instability is still of the thermal concentration nature, but the perturbations with $k \neq 0$ (cellular structure of perturbations) are now the most dangerous. In that region the stability limit increases as R_d is increased (as shown by curve *1*d; the points on curves *1—3* indicate the limits of long-wave and cellular instability). For high R_d the basic flow (1.3) acquires the structure of open boundary layers near walls and virtually stationary main body. Existence in the latter of horizontal temperature and concentration gradients results in the compensation of respective density gradients. Thus at high R_d the problem of flow stability becomes the problem of stability of equilibrium of the vertical layer of mixture with longitudinal stratification.

The analysis of this problem in /8/ showed that instability is related to the development of short-wave convection in the form of stratified flows, with the stability limit and the wave number k_m dependent on R_d

$$G_m = \frac{3.72 R_d^{5/6}}{|P_d - P|}, \quad k_m = 1.03 R_d^{1/6}$$

Data computed for regions of high R_d are in good agreement with these formulas.

The dependence of critical wave numbers k_m of the most dangerous perturbations on R_d is shown in Fig.3, where curves *1—4* correspond to the same values of parameters P and P_d as in Fig.2. The indices *a*, b, c, and d indicate the previously mentioned four instability regions, viz. the hydrodynamic, the concentration wave, the long-wave, and the cellular thermal concentration.

The value of R_{d0} at which the transition from the long-wave $(k_m = 0)$ to cellular $(k_m \neq 0)$ instability takes place in the investigation region of parameters P and P_d is determined, as shown by calculations, by the ratio $P_d / P = \chi / D = S$. The dependence of R_{d0} on S is shown in Fig.4. As shown by this dependence for $P = 0.7$ and $P_d = 1.3$ (gas mixture) the corresponding stability

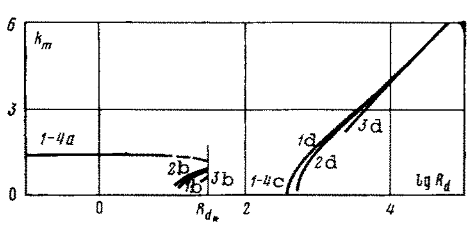

Fig.3

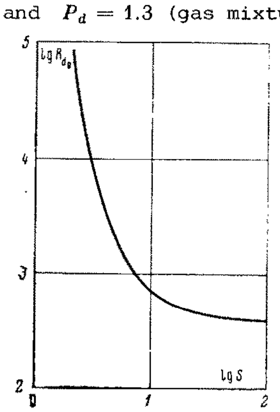

Fig.4

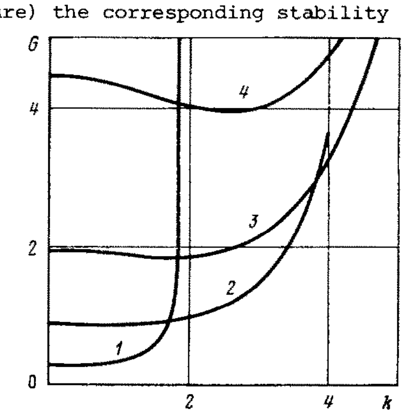

Fig.5

curve *4* in Fig.2 is determined throughout region $R_d > R_{d*}$ by long-wave perturbations.

An example of a set of curves of thermal concentration instability is shown in Fig.5 for $P = 6.7$, $P_d = 676.7$, and various R_d (*1* for $R_d = 64$, *2* for $R_d = 419.4$, *3* for $R_d = 1024$, *4* for $R_d = 2500$). It illustrates the transition of perturbations from the long-wave to the cellular form as R_d is increased.

As mentioned above, the considered here problem was solved in /1/, where the stability limits were numerically determined using the Galerkin method with a basis different from the one used here. The stability limit was determined for $P = 6.7$ and $P_d = 676.7$. The behavior of stability limits determined there for low and high R_d is in agreement with the data in Fig.2 Lowering of the stability limit of thermal concentration origin was also established in /1/. However the results relevant to this most interesting region are erroneous. It should be noted, first of all, that the long-wave mode of thermal concentration instability was not disclosed in /1/, and the destabilization at finite R_d was attributed to cellular perturbations. There are also considerable quantitative discrepancies in the region of the minimum of curve $G_m(R_d)$. Thus for the indicated P and P_d the lowest value of $G_m = 2.1$ is reached, according to /1/ at $R_d = 333$, while formula (2.8) and the numerical results shown in Fig.2 (curve *1*c) yield $G_m = 0.29$ at $R_d = 62.6$. Note that the wave concentration instability mode was not disclosed in /1/.

REFERENCES

1. HART J.E., On sideways diffusive instability. J. Fluid Mech., Vol.49, Pt.2, 1971.
2. TURNER J.S., Double-diffusive phenomena. Annual Rev. Fluid Mech., Vol.6, Palo Alto, Calif., Ann. Reviews Inc., 1974.
3. GERSHUNI G.Z., ZHUKHOVITSKII E. M., and LIUBIMOV D.V., On the stability of stationary convective filtration of a mixture in a vertical porous layer. Izv. Akad.Nauk SSSR, MZhG, No.1, 1980.
4. BAYAZITOGLU Y. and BAYAZITOGLU Y.O., The stability of conduction regime of combined buoyancy mode driven flow in a slot. Lett. Appl. and Eng. Sci., Vol.5, No.4, 1977.
5. GERSHUNI G.Z. and ZHUKHOVITSKII E.M., Convective Stability of Incompressible Fluid. Moscow, "Nauka", 1972.
6. BIRIKH R.V., and RUDAKOV R.N., Application of the orthogonalization method in step-by-step integration in investigations of convective flow stability. In: Hydrodynamics, Issue 5, Uch. Zap. Perm. Univ., No. 316, 1974.
7. GOL'DSHTIK M.A. and SHTERN V.N., Hydrodynamic Stability and Turbulence. Novosibirsk,"Nauka", 1977.
8. THORPE S.A., HUTT P.K. and SOULSBY R., The effect of horizontal gradients on thermohaline convection. J. Fluid Mech., Vol.38, Pt. 2, 1969.

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